Random matrix solution of a polymer collapse model

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Abstract. A polymer folding model on the square lattice is constructed with attractive contact interactions of strength $1/c^2$, $0 < c < 1$. The corresponding model on a dynamical random lattice, with freely fluctuating coordination number at each vertex, is formulated as a random two-matrix model and an expression for the partition function of a length-L chain is derived. Numerical estimates and analytical evaluation for $L \to \infty$ show a third-order collapse transition at $c = \sqrt{2} - 1$. Geometrical critical exponents are computed in each phase and interpreted. The Knizhnik–Polyakov–Zamolodchikov two-dimensional quantum gravity scaling relations are used to predict the corresponding behaviour on the regular lattice, which lies in a different universality class from the percolation Θ-point of Duplantier and Saleur.

1. Introduction

A long polystyrene chain molecule in a solvent undergoes a coil-to-globule transition [1] at a finite temperature $T = \Theta$ due to attractive interactions between different parts of the polymer. Many theoretical techniques have been developed to describe this and other processes of macromolecular folding [2], in particular use has been made of self-interacting random walks on lattices such as the honeycomb lattice [3] and Sierpinski fractals [4]. In this paper a model consisting of folding chains with contact interactions is constructed on the two-dimensional square lattice. In order to study the model it is reformulated on the ensemble of two-dimensional simplicial lattices with random fluctuations of the local intrinsic curvature, which appear in the study of two-dimensional quantum gravity [5, 6]. Statistical mechanics on lattices in this ensemble represents a dynamical (annealed rather than quenched) average over a certain class of fractals. Although an apparently more complicated problem, the advantage is that such statistical mechanics problems are often exactly solvable using random matrix models [6], which yield expressions for the thermodynamic and some of the geometrical quantities of interest. The results obtained can be used to infer the corresponding behaviour on a regular two-dimensional lattice, since in all of the many solved examples the qualitative phase structure of such statistical systems is the same. Moreover, when the system is conformally invariant on the regular lattice, there are well known techniques for relating critical exponents to those of the system on the fluctuating lattice [7].

The organization and main results of this paper are as follows. In the next section the polymer model is defined on a square lattice and its relation to other models which use a different microscopic defintion of contact is noted. The dynamical random square lattice is then introduced. Section 3 explains the random matrix representation and section 4 describes the standard procedure for solving this matrix model using orthogonal polynomials. In section 5 the properties of a single polymer are studied, although the construction in

sections 3 and 4 is valid for a finite polymer fugacity, establishing the exact temperature $T = T_c$ of a third-order collapse transition. By expanding about $T = T_c$ and $T = \infty$, the supposedly universal exponent a in the asymptotic number of configurations of a length $L \to \infty$ polymer, $\sim L^a e^{bL}$, is determined to be $a = -5/2$ for $T \to \infty$ and $T \to T_c^+$ (and presumably for all $T > T_c$) while $a = -3/2$ for $T = T_c$. The asymptotic behaviour in the $T < T_c$ phase is more subtle but a value $a = -1$ is obtained for $T \to T_c^-$. An attempt to understand the charateristic geometrical behaviour implied by these results is made in section 6 by calculation of the dimensions of the scaling polymer 'star' operators. The phase structure and scaling dimensions on a fixed regular lattice are given using the Knizhnik–Polyakov–Zamolodchikov (KPZ) scaling relation [7]. It is concluded that $T > T_c$ and $T = T_c$ correspond to the standard dilute and dense [8] phases respectively of a self-avoiding walk in the plane. The case $T < T_c$ is less clear but there is evidence that it entails an ultra-compact phase whereby the lattice is filled with both polymer and contact points. In any case the collapse transition appears to be in a different universality class from the percolating cluster Θ-point in [3]; the microscopic definition of contact used in this paper seems to cause a rather more severe collapse. Conclusions are summarized in section 7 and a simple co-polymer model with random sequencing of $n(\psi)$ and $n(\phi)$ monomers of types ψ and ϕ respectively, together with a fugacity for the ratio $n(\phi)/n(\psi)$ is formulated in an appendix and solved with no extra work.

2. Polymer model

The polymer model is first defined on a fixed square lattice. A polymer chain of L steps is a random walk on the sides (links) of squares which cannot cross itself (excluded volume), but can occupy a given link any number of times. Examples of allowed and disallowed configurations are illustrated in figure 1. Two or more steps of the polymer ocuppying the same link of the square lattice incur a contact interaction. The single polymer partition function is the sum over all configurations of the polymer of length L with a weight for multiple occupation of each occupied link i given by $c^{2-2w(i)}$, where $w(i)$ is the number of steps occupying link i and $0 < c \leqslant 1$ is a an attractive contact coupling. Note that this defintion of contact differs from those often used in studies of self-avoiding random walks which cannot use a given link more than once. The one employed here is certainly a less physical model of the steric repulsion of polymers but leads to an analytically tractable random matrix theory later.

For the fixed regular square lattice each vertex is surrounded by $s = 4$ squares. If $s < 4$ ($s > 4$) at a vertex the two-dimensional lattice would be intrinsically curved at that vertex, with positive (negative) Gaussian curvature. For the dynamical random planar lattice s is allowed to be a freely fluctuating independent variable at each vertex, subject only to a fixed total number of A squares in the surface say. In other words the ensemble of random fluctuating lattices is obtained by gluing pairwise along links, A squares each of link length δ say, in all possible ways so as to form an *abstract* surface with spherical topology say (figure 2). A plays the role of an infrared cutoff on the size of the lattice. The polymer is a random walk on links of squares as before. More generally one can introduce chemical potentials Γ and g conjugate to the number of polymers n and number of squares A respectively. After multiplying by c^{2L}, the partition function is

$$\mathcal{Z} = \sum_{\tilde{w}, n, A} c^{2L - \tilde{w}} \Gamma^n g^A H(A, \tilde{w}, n, L) \tag{1}$$

where $\tilde{w} = \sum_i (2w(i) - 2)$ and H is the number of configurations of n polymers, each of length L, with $\tilde{w}/2$ contacts, on all possible surfaces made from A squares with spherical
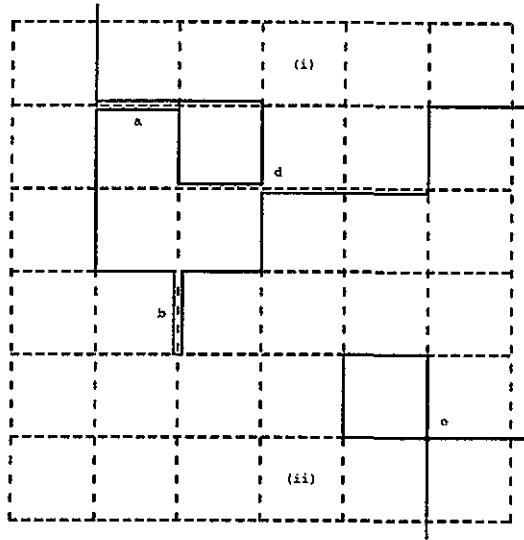
Figure 1. (i) Allowed configurations of the polymer on a square lattice. The double occupation of a link at 'a' and 'b' for example incurs a contact weight $1/c^2$. Point 'd' is not counted as a contact. (ii) The polymer is not allowed to cross itself, as at 'e'.
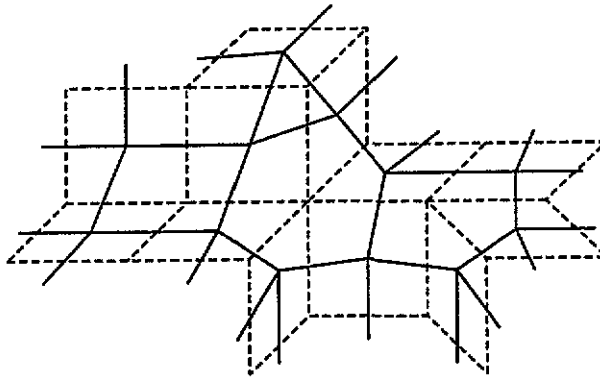

Figure 2. A portion of a two-dimensional random square lattice (dotted lines). The full lines construct the dual graph given by the Feynman diagram expansion of the random matrix model.

topology. If one imagines L bonds along each side of the polymer which either connect it to a lattice square or directly to another part of the polymer, with the normalization of Z chosen in equation (1) these have polymer–lattice E_{pl} or polymer–polymer (contact) E_{pp} bond energies corresponding to the Boltzmann weights at temperature T

$$\exp(-E_{pl}/T) = c \qquad \exp(-E_{pp}/T) = 1. \tag{2}$$

Thus $E_{pp} = 0$ and, if T is measured in units of E_{pl}, one has the correspondence $T = -(\log c)^{-1}$; hence $c \to 1$ is the high-temperature and $c \to 0$ the low-temperature limit. So far all the geometrical construction has been discrete but it turns out that H grows as $\sim (1/g_c)^A$ and therefore there exists a $g = g_c$ at which large-area A surfaces become critical and a universal continuum limit can be established.

3. Matrix model

In the infinite temperature case $c = 1$, this problem was mapped onto a random matrix model by Duplantier and Kostov [9]. In order to analyse $0 < c < 1$ a more general random matrix model will be employed which is, however, still solvable. Let us first recall how the random surfaces described earlier can be generated from random matrices†. Consider the partition function

$$z = \int \prod_{i=1}^{N} \mathrm{d}\phi_{ii} \prod_{i>j} \mathrm{d}\phi_{ij}\, \mathrm{d}\phi_{ij}^{*} \exp\left(\mathrm{Tr}\left[-\frac{1}{2}\phi^2 + \frac{g}{4N}\phi^4 \right] \right) \tag{3}$$

where ϕ is an $N \times N$ Hermitian matrix with elements ϕ_{ij}. If one expands the integrand in g and performs the Gaussian integrals, the expansion can be given a diagrammatic representation (figure 2) of quartic vertices tied together by the propogators to form a closed graph, all possible graphs with A vertices contributing at order g^A. Note that we are interested only in connected graphs, so we should take the logarithm of z. Because ϕ is an $N \times N$ matrix each graph is also weighted by a power of N. One finds

$$\log z = \sum_{\mathcal{G}} g^A \frac{N^{Q-A}}{C(\mathcal{G})} \tag{4}$$

where the sum is over all connected closed graphs $\mathcal{G}$ of coordination number four. Q is the number of loops in the graph, while $C(\mathcal{G})$ is the order of the symmetry group of $\mathcal{G}$. These graphs, shown by full lines in figure 2, are the dual graphs corresponding to the squares glued along links described earlier without any restriction on topology of the closed surface formed. The genus G of this surface is defined by Euler's relation $v - A = 2 - 2G$ where A is the number of square faces and v the number of vertices in the random square graph $\tilde{\mathcal{G}}$. Since by duality $Q = v$ and $C(\tilde{\mathcal{G}}) = C(\mathcal{G})$, the weight for a genus G surface is N^{2-2G}. As $N \to \infty$, surfaces of spherical topology $G = 0$ dominate and this is the limit which will always concern us in this paper.

In order to introduce interacting polymers on the surfaces consider two random Hermitian $N \times N$ matrices ϕ and ψ each with the previous measure (3) and the following partition function

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \exp\left(\mathrm{Tr}\left[-\tfrac{1}{2}\phi^2 - \tfrac{1}{2}\psi^2 + c\phi\psi + \frac{g(1-c^2)^2}{4N}\phi^4 + \frac{(1-c^2)^L}{2}\Gamma\psi^{2L} \right] \right). \tag{5}$$

The ϕ matrix generates random surfaces as before, but ψ introduces a new vertex type ψ^{2L} with coordination number $2L$ and two new propogators in addition to ϕ^2, which tie together vertices of the same type (using ϕ^2 or ψ^2) or of different types (using $\psi\phi$) (figure 3(i)). The propogator weights are

$$\left\langle \frac{1}{N} \mathrm{Tr}\, \phi^2 \right\rangle = \left\langle \frac{1}{N} \mathrm{Tr}\, \psi^2 \right\rangle = \frac{1}{1-c^2} \qquad \left\langle \frac{1}{N} \mathrm{Tr}\, \psi\phi \right\rangle = \frac{c}{1-c^2}. \tag{6}$$

The weights have been chosen so that in any graph the factors of $(1-c^2)$ cancel completely between the propogators and vertices, so ϕ^4 vertices are weighted by g and ψ^{2L} by ΓL. Forming the dual graphs once again, the ψ^{2L} vertices are interpreted as holes in the surface of length $2L$. The final step in visualizing the polymers is to use the trick in [9] whereby each whole of length $2L$ in the dual graph is sewn up starting at a given point on the edge of the hole to form a seam (polymer) of length L, illustrated in figure 3(ii). The extra factor of L in the weight of a hole is equivalent to the L possible positions of the given point

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† A review of this technique and methods of solution can be found in [10] for example.

which yield a distinct polymer/surface configuration. Note that the propagator $\psi\phi$ acts to connect the polymer to the lattice with weight c while the propgator ψ^2 acts to connect one part of the polymer directly to another part with contact weight one (cf (2)). The $N \to \infty$ limit of $\log Z$, equation (5), then generates the required partition function $\mathcal{Z}$ (equation (1)) on genus zero surfaces.
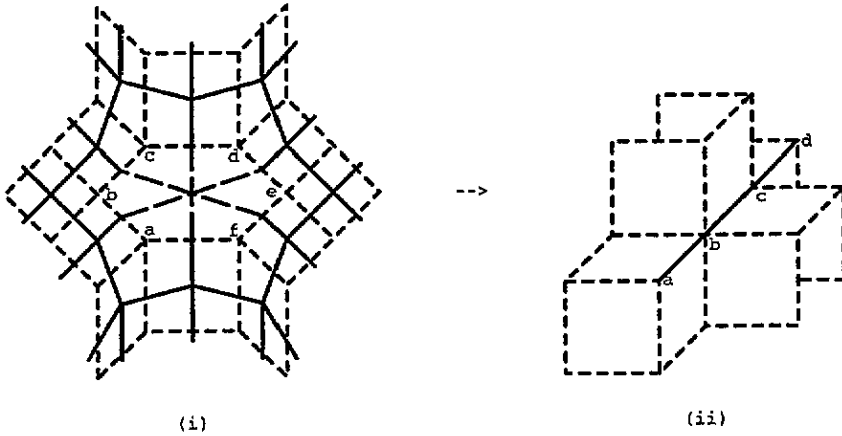

(i)   (ii)

Figure 3. (i) A portion of a two-dimensional random square lattice with a hole of length 6 dual to the vertex created by ψ^{2L} in the particular case $L = 3$. In the dual graph lines associated with ϕ are full while lines associated with ψ are chain lines. (ii) Sewing up the hole in the random square lattice to form a seam representing a polymer. The seam could begin at 'a', 'b', or 'c' to give distinct polymer/lattice configurations. The case shown corresponds to sewing 'b' to 'f' and 'c' to 'e'.

4. Orthogonal polynomial solution

The matrix model (5) can be solved in the large-N limit by the method of orthogonal polynomials [11]. Using the decomposition of Hermitian matrices into unitary and diagonal matrices, the integration over the unitary matrices can be explicitly performed to leave

$$Z = \int \prod_{i=1}^{N} \mathrm{d}\phi_i \, \mathrm{d}\psi_i \, \Delta(\phi)\Delta(\psi) \exp\left(-\sum_{i=1}^{N} V(\phi_i, \psi_i) \right) \tag{7}$$

where

$$\Delta(\phi)\Delta(\psi) = \prod_{i>j} (\phi_i - \phi_j)(\psi_i - \psi_j) \tag{8}$$

and

$$V(\phi, \psi) = \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\psi^2 - c\phi\psi - \frac{g(1-c^2)^2}{4N}\phi^4 - \frac{(1-c^2)^L}{2}\Gamma\psi^{2L} \tag{9}$$

with ϕ_i and ψ_i the eigenvalues of ϕ and ψ. Introducing polynomials of degree i and j with first coefficient normalized to unity: $P_i(\phi) = \phi^i + \cdots$, $Q_j(\psi) = \psi^j + \cdots$, where ϕ and ψ are now understood as real variables parametrizing the eigenvalues of the matrices, they are defined to be orthogonal under

$$\int_{-\infty}^{+\infty} \mathrm{d}\phi \, \mathrm{d}\psi \, \mathrm{e}^{-V(\phi,\psi)} P_i(\phi) Q_j(\psi) = h_i \delta_{ij}. \tag{10}$$

Using this property one can show that [11]

$$\log Z = \text{constant} + \sum_{i=1}^{N-1} (N-i) \log f_i \tag{11}$$

where $f_i = h_i/h_{i-1}$ and the constant is g-independent. P and Q satisfy recursion relations

$$\phi P_i(\phi) = P_{i+1} + r_i P_{i-1} + s_i P_{i-3} + \cdots \tag{12}$$

$$\psi Q_j(\psi) = Q_{j+1} + q_j Q_{j-1} + t_j Q_{j-3}. \tag{13}$$

The one for P terminates after $L+1$ terms. One can find a set of equations for r, s, q, t, and f which can be solved for f, and thus $\log Z$. The required equations are given by the identities

$$\int_{-\infty}^{+\infty} d\phi \, d\psi \, e^{-V(\phi,\psi)} P'_{i-1} Q_i = 0 \tag{14}$$

$$\int_{-\infty}^{+\infty} d\phi \, d\psi \, e^{-V(\phi,\psi)} (P'_i - i\phi^{i-1}) Q_{i-1} = 0 \tag{15}$$

$$\int_{-\infty}^{+\infty} d\phi \, d\psi \, e^{-V(\phi,\psi)} P'_{i-3} Q_i = 0 \tag{16}$$

and their counterparts obtained by exchanging $P \leftrightarrow Q$ and $\phi \leftrightarrow \psi$. By integrating by parts they are explicitly found to be

$$cq_i = f_i \left[1 - \frac{g(1-c^2)^2}{N} (r_{i+1} + r_i + r_{i-1}) \right] \tag{17}$$

$$i = r_i - cf_i - \frac{g(1-c^2)^2}{N} [s_i + s_{i+1} + s_{i+2} + r_i(r_{i+1}r_i + r_{i-1})] \tag{18}$$

$$ct_i = -\frac{g(1-c^2)^2}{N} f_i f_{i-1} f_{i-2} \tag{19}$$

$$cr_i = f_i - \Gamma \sqrt{f_i} \theta[i, i-1] \tag{20}$$

$$i = q_i - cf_i - \Gamma \sqrt{f_i} \theta[i-1, 1] \tag{21}$$

$$cs_i = -\Gamma \sqrt{f_i f_{i-1} f_{i-2}} \theta[i, i-3] \tag{22}$$

where

$$\theta[i, j] = L(1-c^2)^L \int d\phi \, d\psi \, e^{-V(\phi,\psi)} \frac{P_i}{\sqrt{h_i}} \psi^{2L-1} \frac{Q_j}{\sqrt{h_j}}. \tag{23}$$

In the large-N limit one can define a continuous variable $x = i/N$ and functions $f_i = Nf(x)$, $r_i = Nr(x)$, $q_i = Nq(x)$, $s_i = N^2s(x)$, and $t_i = N^2t(x)$ so that the previous equations become

$$cq = f - 3g(1-c^2)^2 fr \tag{24}$$

$$x = -cf + r - g(1-c^2)^2(3s + 3r^2) \tag{25}$$

$$ct = -g(1-c^2)^2 f^3 \tag{26}$$

$$cr = f - \Gamma \sqrt{f} \theta[i, i-1] \tag{27}$$

$$x = -cf + q - \Gamma \sqrt{f} \theta[i-1, i] \tag{28}$$

$$cs = -\Gamma f^{3/2} \theta[i, i-3]. \tag{29}$$

Equations (29) and (25) will be redundant in this paper, while substituting equations (27) and (24) into equation (28) gives an equation for f

$$x = -cf + \frac{f}{c} - \frac{3g(1-c^2)}{c^2}(f^2 - \Gamma f^{3/2} \theta[i, i-1]) - \Gamma \sqrt{f} \theta[i-1, i]. \tag{30}$$

So far the large-N limit has been taken in order to isolate the random surfaces with genus zero, but g is still a free parameter conjugate to the area A. The perturbation series of equation (11) in g is actually only convergent for $g < g_c$, which is a reflection of the fact that the number of configurations H in equation (1) grows as $(1/g_c)^A$. Therefore by tuning $g \to g_c$ the infinite-A surfaces become critical in the partition function and a universal continuum limit is attained [6] (universal in the sense that one could equally have discretized the surfaces with some other type of polygon instead of squares). To identify this critical point of the surfaces in the orthogonal polynomial formalism it is expedient to set $\Gamma = 0$ (no polymers) and identify the singularity in the solution for $f(1) = f$ of equation (30) as g is varied:

$$1 = -cf + \frac{f}{c} - \frac{3g(1 - c^2)^2 f^2}{c^2} \tag{31}$$

yielding

$$g_c = \tfrac{1}{12} \qquad f_c = \frac{2c}{1 - c^2}. \tag{32}$$

As $g \to g_c$ it is useful to define a renormalized (physical) area of the surfaces; if $g = g_c(1 - \mu\delta^2)$, where $\delta \to 0$ is the link length of squares, μ is a renormalized variable conjugate to renormalized surface area $\mathcal{A} = A\delta^2$. $\mathcal{A}$ plays the role of infrared cutoff on the area of continuum surfaces for which the ultraviolet cutoff δ has now been removed. In the continuum limit $\delta \to 0$, μ not only couples to the universal continuum surfaces ($\mathcal{A}$ finite) but also to the surfaces finite in lattice units A ($\mathcal{A} \sim O(\delta) \to 0$). The partition function (11) will have a part regular in μ, usually representing the latter surfaces, and a non-analytic part representing the former. The thermodynamic limit is of course the infinite area one, $\mathcal{A} \to \infty$. Now at large N one has from equation (11)

$$\log Z = N^2 \int_0^1 \mathrm{d}x\,(1 - x) \log f(x) + \text{constant.} \tag{33}$$

The singular behaviour of $\log Z$ arises from $x \to 1$ so it is appropriate to define scaling variables

$$x = 1 - z\delta^2 \qquad f(x) = f_c(1 - u(z)\delta). \tag{34}$$

Then the first non-vanishing order in δ in equation (30) occurs at $O(\delta^2)$ giving (at $\Gamma = 0$)

$$u^2 = \mu + z. \tag{35}$$

Hence [6]

$$\log Z_{\Gamma=0} \equiv Z_0 = -N^2\delta^5 \int_0^\infty \mathrm{d}z\, zu + \text{regular}$$

$$= N^2\delta^5 \tfrac{4}{15}\mu^{5/2} + \text{regular.} \tag{36}$$

If u is a function of the combination $\mu + z$ only, as above, then $N^2\delta^5 u(\mu) = -\partial^2 \log Z/\partial\mu^2$ and it has the interpretation of a 'susceptibility'. In the presence of polymers equation (35) is modified by a Γ-term; in this paper only the case of a single polymer will be addressed in detail.

5. Single polymer collapse

For a single polymer equation (35) is still appropriate and one studies the connected expectation value of $\mathrm{Tr}(\psi^{2L})$ in this theory, i.e. $\log Z$ expanded to first order in Γ,

$\log Z = Z_0 + \Gamma Z_1 + \cdots$. In terms of orthonormal polynomials, connected one-point Green's functions of the matrix model are given by

$$Z_1 = Z_0 \langle (1 - c^2)^L \, \mathrm{Tr}(\psi^{2L}) \rangle_{\Gamma=0}$$

$$= (1 - c^2)^L \sum_{i=1}^{N} \int d\phi \, d\psi \, e^{-V(\Gamma=0)} \frac{P_i}{\sqrt{h_i}} \psi^{2L} \frac{Q_i}{\sqrt{h_i}}. \tag{37}$$

In order to evaluate the last quantity it is illuminating to introduce raising and lowering operators on the orthonormal polynomials defined by

$$a^\dagger \left[\frac{Q_i}{\sqrt{h_i}} \right] = \left[\frac{Q_{i+1}}{\sqrt{h_{i+1}}} \right] \qquad a \left[\frac{Q_i}{\sqrt{h_i}} \right] = \left[\frac{Q_{i-1}}{\sqrt{h_{i-1}}} \right]. \tag{38}$$

Then from (13)

$$\psi \equiv \sqrt{f} a^\dagger + \frac{q}{\sqrt{f}} a + \frac{t}{f^{3/2}} a^3 \tag{39}$$

$$cq = f \left(1 - \frac{3g(1-c^2)^2 f}{c} \right) \qquad ct = -g(1-c^2)^2 f^3. \tag{40}$$

Thus by orthogonality, ψ^{2L} contributes in (37) only when there are equal numbers of a and $a^\dagger$ in its expansion. Defining $Z_1 = \sum_{i=1}^{N} Z_1'$, one finds (as always, for $N \to \infty$)

$$Z_1'(L, c, f, g) = (1 - c^2)^L (2L)! f^L \sum_{p=0}^{[L/2]} \frac{[c - 3g(1-c^2)^2 f]^{L-2p} [-g(1-c^2)^2 f]^p c^{3p-2L}}{(L - 2p)!(p + L)!p!} \tag{41}$$

$$\frac{(1-c^2)^2 gf}{c} = \frac{1-c^2}{6} [1 - u\delta + O(\delta^2)]$$

where u is given by equation (35) and $[L/2]$ indicates greatest integer $\leqslant L/2$. This formula is the first main result.

If, as the continuum limit $\delta \to 0$ is taken, L remains finite, the polymer will appear merely as an infinitesimal puncture of length $2L$ in the surfaces of finite renormalized area $\mathcal{A}$. In this case equation (41) reduces to the form

$$Z_1 = N \int_0^\infty dz \, [X(L, c) - Y(L, c)u\delta + O(\delta^2)]. \tag{42}$$

The parts regular in μ, including the first term, represent the non-universal contributions due to the polymer on finite-$\mathcal{A}$ surfaces. The non-analytic part at lowest order in δ gives the scaling behaviour of the puncture operator one-point function on the finite-$\mathcal{A}$ surfaces in the continuum limit

$$Z_1 = \text{regular} + \tfrac{2}{3} N \delta^3 Y(L, c) \mu^{3/2}. \tag{43}$$

This is basically $\partial Z_0 / \partial \mu$ (cf equation (36)) since a puncture can be anywhere on the surface and therefore measures its area $\mathcal{A}$ conjugate to μ. Note that the overall power of N occurs because Euler's relation introduced earlier is more generally $v - A = 2 - 2G - h$ where h is the number of holes in the surface. Changing L or c does not change the scaling dimensions and there are no singularities of $Y(L, c)$, so no phase transitions.

In order that the polymer remain an extended object on the surfaces of finite $\mathcal{A}$ one must scale its length as $L = l\delta^{-2/\nu D}$, for some appropriate exponent νD. D is understood as some intrinsic fractal dimension of the lattice which would be two if the lattice were regular, while ν is the usual mean square size exponent of a polymer. In order to find νD, and in particular its dependence upon c, one must evaluate the large-L behaviour of the sum (41). This is not straightforward because the terms in the sum alternate in sign. For
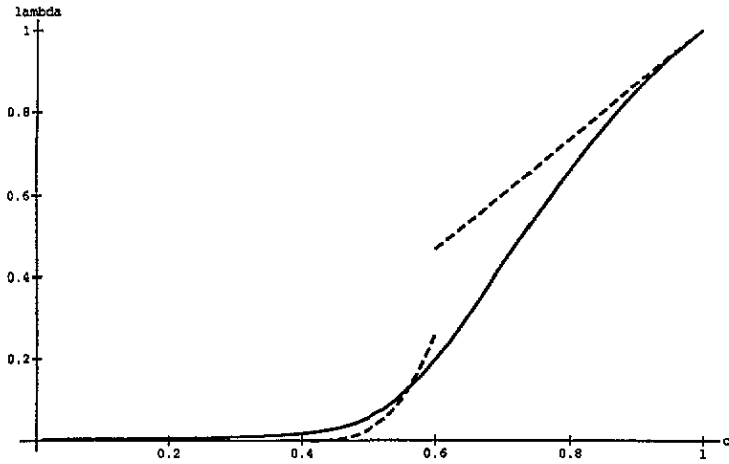
Figure 4. The induced free energy per unit length $u\lambda$ estimated from equations (41) and (46) at $L = 150$. Broken lines are the leading perturbative results around $c = 1$ and $c = \sqrt{2} - 1$ for $L = \infty$.

the case $c = 1$ [9], which corresponds to infinite temperature, only the first term survives, leading to the result $\nu D = 1$ and

$$Z'_1 = \frac{2^L (2L)!}{(L!)^2} \exp(-ul) + \text{regular} \tag{44}$$

in the continuum limit $\delta \to 0$. The large-L behaviour of the prefactor is $\sim 2^L/\sqrt{L}$, showing the characteristic (non-universal) exponential growth of the number of configurations, and that for Z_1 the part non-analytic in μ grows as $\sim 2^L/L^{5/2}$. The exponent $5/2$ is expected to be universal and one of the main objectives is to find its dependence upon c. Implicitly we are therefore assuming that this characteristic form of the large-L behaviour persists as c is reduced from one, i.e. when $L = l/\delta$

$$Z'_1 = G(L, c)[e^{-\lambda(c)ul + O(\delta)} + O(\text{constant}^{-O(\delta^{-1})})] \tag{45}$$

where $G \propto L^a e^{b(c)L}$ and a is universal. As always, by universal it is meant independent of the form of discretization of the surface, rather than independent of the microscopic definition of contact interactions. The author has not managed to *derive* the asymptotic form (45) except in the neighbourhoods of $c = 1$ and $c = \sqrt{2} - 1$, which will be discussed shortly.

It is, however, useful for orientation to first investigate the exact formula (41) numerically at large but finite L and fit it to the form (45). Evaluation of (41) for increasing L indeed confirms exponential growth with L for general c. Figure 4 plots $\lambda(c)$ obtained from

$$ul\lambda = \log\left[\frac{Z'_1(L, c, 0)}{Z'_1(L, c, ul)}\right] \tag{46}$$

where Z'_1 is evaluated for $L = 150$—corrections to the RHS above are $O(\delta) \sim O(1/L)$. $b(c)$ does not depend upon μ and so represents some short-distance lattice artifacts. On the other hand $u\lambda$ is expected to be universal and represents an induced free energy per unit length of the polymer due to fluctuations of the finite-$\mathcal{A}$ surfaces†; it is non-analytic

† Note that it is the presence of an extra dimensionful parameter μ on the fluctuating lattice which enables one to extract a universal part from the free energy of the polymer, which, in the case of a dilute polymer, is like a boundary free energy.

in μ from equation (35). The behaviour of $\lambda(c)$ thus governs the interesting geometrical properties of the polymer. In particular, figure 4 seems to show a transition to $\lambda(c) = 0$ at $c \sim 0.4$, which will be verified shortly. Physically, as c is reduced the polymer favours polymer–polymer rather than polymer–lattice bonds and the induced free energy λ falls. Below a certain $c_c \approx 0.4$ the polymer collapses entirely and is only connected into the surface by a small number of polymer–lattice bonds. This is the coil-to-globule transition on fluctuating random surfaces. The polymer–polymer contacts also induce free energy per unit length but at a lower order in δ, and it will be necessary to choose a different scaling exponent νD in order to see this contribution near $c = c_c$.

In order to find c_c exactly, determine the exponents a and νD, and the details of the purported phase transition, it is useful to make the replacements $a \to e^{ip}, a^\dagger \to e^{-ip}$ in ψ (39), enforcing the cancellation of a's and $a^\dagger$'s by integrating over p thus (cf equations (37) and (39))

$$Z_1' = (1 - c^2)^L \int_{-\pi}^{\pi} \frac{dp}{2\pi} \left[\sqrt{f} e^{-ip} + \frac{q}{\sqrt{f}} e^{ip} + \frac{t}{f^{3/2}} e^{3ip} \right]^{2L}. \tag{47}$$

As the continuum limit $\delta \to 0$, $L \to \infty$, is approached one would like the integrand to exponentiate simply using the property $(1 - \beta/L)^{2L} \to e^{-2\beta}$ (β finite). To see why this does not happen in general it is expedient to expand ψ in δ and p, using previous definitions for f, q, and t in terms of scaling variables u and μ;

$$\psi \equiv \frac{1}{\sqrt{(1 - c^2)2c}} [a_1 + a_2 ip + a_3 u + a_4 p^2 + a_5 iup\delta + a_6 u^2\delta^2 + a_7 \mu\delta^2$$
$$+ a_8 ip^3 + a_9 up^2\delta + a_{10} iup^3\delta + a_{11} p^4 + \cdots]. \tag{48}$$

The coefficients $a_i(c)$ are easily computed, in particular

$$a_1 = \tfrac{1}{3}(3 + 5c + 3c^2 + c^3) \tag{49}$$
$$a_2 = 1 - 3c + c^2 + c^3 \tag{50}$$
$$a_3 = \tfrac{1}{2}(1 - c - 3c^2 - c^3) \tag{51}$$
$$\cdots \quad \cdots \text{etc.}$$

This and subsequent algebraic computations were done by hand and checked by computer. Since $a_2 \neq 0$ in general, this gives the β above an (infinite) imaginary part and the resultant oscillatory behaviour means that one must keep all orders of the expansion in p (48) in order to even begin to evaluate the integral (47). This problem will turn out to be obviated if $a_2 = 0$ for some value of c since one can then easily separate off the contribution to the p-integral non-analytic in μ, coming from the $p \sim O(\delta^{1/\nu D})$ region. In the interval $0 \leqslant c \leqslant 1$, a_2 has roots at $c = 1$ and $c = \sqrt{2} - 1$, the latter shortly being interpreted as the collapse temperature c_c. The strategy now is to expand c about these two roots.

First consider the neighbourhood of $c = 1$ (high temperature limit). In order to reproduce the $c = 1$ solution [9] one finds that for $L = 1/\delta$ ($\nu D = 2$) the contribution of the integral (47) non-analytic in μ comes from the region $p \sim O(\sqrt{\delta})$. Writing $c = 1 - \epsilon$ for small ϵ, $a_2 = O(\epsilon)$ and working perturbatively to lowest order in ϵ yields

$$\psi^{2L} \equiv ((1 - c^2)2c)^L \left[a_1 + \left(a_3 + \frac{a_2 a_5}{2a_4} \right) u\delta + \left(a_4 + \frac{3a_2 a_8}{2a_4} \right) \tilde{p}^2\delta + \cdots \right]^{2L} \tag{52}$$

$$\sqrt{\delta}\tilde{p} = p + i\sqrt{\delta}\rho \qquad \rho = \frac{a_2}{2a_4\sqrt{\delta}} + \frac{a_5 u\sqrt{\delta}}{2a_4} + O(\epsilon^2, \epsilon u\sqrt{\delta})$$

where ellipses indicate either $O(\delta^{3/2})$ terms, which do not survive the continuum limit, or terms which renormalize the displayed coefficients in (52) at $O(\epsilon^2)$. Therefore

$$Z_1' = \frac{a_1^{2L}\sqrt{\delta}}{2\pi(2c)^L} \int_{i\rho-\pi/\sqrt{\delta}}^{i\rho+\pi/\sqrt{\delta}} d\tilde{p} \, \exp\left[2l\left(\frac{a_3}{a_1} + \frac{a_2 a_5}{2a_1 a_4}\right)u + 2l\left(\frac{a_4}{a_1} + \frac{3a_2 a_8}{2a_1 a_4}\right)\tilde{p}^2 + \cdots\right] \quad (53)$$

$$= \sqrt{\frac{\delta}{4\pi l}}[8 + O(\epsilon)]^L \exp\left[-ul\left(1 - \frac{4\epsilon}{3} + O(\epsilon^2)\right)\right] + \text{regular} \quad (54)$$

in the continuum limit $\delta \to 0$, where the integral

$$\lim_{\delta \to 0} \int_{i\rho-\pi/\sqrt{\delta}}^{i\rho+\pi/\sqrt{\delta}} dx \, e^{-x^2} = \sqrt{\pi} \qquad \lim_{\delta \to 0} \sqrt{\delta}\rho < \pi \quad (55)$$

has been used, valid since $\rho\sqrt{\delta} \sim O(\epsilon) \ll 1$ by hypothesis. The same result can also be derived by expanding the formula (41) in ϵ. Equation (54) is of the form (45) with a λ that correctly matches onto the $L = 150$ curve of figure 4 near $c = 1$. The partition function itself would then be

$$Z_1 = N\delta^2 \int_0^\infty dz \, Z_1' \quad (56)$$

$$\propto N\delta^{5/2}\frac{\text{constant}^L}{\lambda l^{3/2}}\left(\sqrt{\mu} + \frac{1}{\lambda l}\right)e^{-\lambda l\sqrt{\mu}} + \text{regular} \quad (57)$$

using equation (35). Large-area $\mathcal{A} \to \infty$ corresponds to $\mu \to 0$, in which case the power exponent is $a = -5/2$.

Fortunately there is one other value of c where a_2 vanishes, given by $c = \sqrt{2} - 1$; a_3 also vanishes at this point. Writing $c = \sqrt{2} - 1 + \epsilon$ for small ϵ

$$a_2 = 4(1 - \sqrt{2})\epsilon + (3\sqrt{2} - 2)\epsilon^2 + \epsilon^3 + O(\epsilon^4)$$

$$a_3 = -2\epsilon - \frac{3\sqrt{2}\epsilon^2}{2} - \frac{\epsilon^3}{2} + O(\epsilon^4).$$

All other a_i's in (48) are non-vanishing at $c = \sqrt{2} - 1$ and it is straightforward to evaluate their ϵ-expansions and substitute in (48) once again. The result to lowest non-zero order in ϵ, which requires one to consider all terms displayed in equation (48), is after a tedious calculation

$$\psi^{2L} = \left(\frac{a_1}{\sqrt{(1 - c^2)2c}}\right)^{2L} \exp[-\alpha\epsilon^3 ul - (2a_4/a_1)\tilde{p}^2 l + \cdots] \quad (58)$$

where ellipses are $O(\delta^{1/2})$ terms which do not contribute to the continuum limit or terms which renormalize the displayed coefficients beyond the leading non-zero order in ϵ. $2a_4/a_1 = 9 + O(\epsilon)$ while α to lowest order in ϵ is given by

$$\alpha\epsilon^3 = \epsilon\left(\frac{a_2 a_5}{a_4 a_1} + 2\frac{a_3}{a_1}\right) - \epsilon^2\left(\frac{3a_8 a_2^2 a_5}{4a_4^3 a_1} + \frac{a_9 a_2^2}{2a_4^2 a_1}\right)$$

$$-\epsilon^3\left(\frac{a_{10}a_2^3}{4a_4^3 a_1} - \frac{a_{11}a_5 a_2^3 a_5}{2a_4^4 a_1} + \frac{3a_8 a_2^2}{8a_4^5 a_1}(3a_8 a_2 a_5 + 2a_4 a_2 a_9)\right) + O(\epsilon^4). \quad (59)$$

Numerically $\lim_{\epsilon \to 0} \alpha \approx 35$. Once again the appropriate scaling behaviour was $L = l/\delta$. Performing the $\tilde{p}$-integral and using (35) the partition function is then

$$Z_1 \propto \text{constant}^L N\delta^{5/2}\frac{1}{\sqrt{l}}\int_\mu^\infty dy \, e^{-\alpha\epsilon^3\sqrt{y}l} + \text{regular} \quad (60)$$

$(u = \sqrt{\mu + z} \equiv \sqrt{y})$ which is again of the form (54) with

$$\lambda = \alpha\epsilon^3 + O(\epsilon^4). \quad (61)$$

The calculation for large but finite L shown in figure 4 approaches this limiting behaviour with increasing L. At $\epsilon = 0$ there is a collapse transition because as $\lambda \to 0$

$$Z_1 \to \text{regular} + \tfrac{1}{3}N\delta^{5/2}\text{constant}^L\lambda\sqrt{l}\mu^{3/2} \tag{62}$$

which is of the same form as the puncture operator one-point function (42)—the polymer has collapsed to a point. Since $\lambda\sqrt{\mu}$ represents the (non-analytic part of the) free energy per unit length of the polymer due to its connections into the lattice, its vanishing (61) can be interpreted as a third-order collapse transition if λ is identically zero for $c < \sqrt{2} - 1$. (This is consistent with the evaluation of the exact result (41) for $L = 150$ in figure 4.) For negative ϵ the integral for the partition function (60) is not convergent. This is because the higher orders in δ have been neglected, i.e. there are corrections to the argument of the exponential integrand $ly\delta, ly^{3/2}\delta^2$, etc which become important at $y \sim O(1/\delta^2)$, contributing to the regular part. In terms of the discrete variable used earlier, $i = Nx$, the sum $Z_1 = \sum_{i=1}^{N} Z_1^i$ no longer has a singular part in μ coming from the finite-i region but only the regular part coming from $i \sim O(N)$. In fact the result of the y-integral becomes, to leading order in δ, some μ-independent constant since it is insensitive to the lower limit and μ enters explicitly in the argument of the exponential only at lower order in δ (terms like $l\mu\delta, l\mu\sqrt{y}\delta^2, \ldots$). Therefore $\lambda = 0$ for $c = \sqrt{2} - 1 + \epsilon$, perturbatively in $\epsilon < 0$.

In order to make the polymer an extended object once again when $c \leqslant \sqrt{2} - 1$ a different exponent νD must be chosen. Consider first the collapse point itself, $c = \sqrt{2} - 1$. The order u term in ψ (48) vanishes and one must consider the next order in δ, that is μ and u^2 terms, which gives

$$\psi^{2L} \equiv \left(\frac{a_1}{\sqrt{(1-c^2)2c}}\right)^{2L} \exp[(\mu - 3u^2/2)l - 9\bar{p}^2 l + \cdots]. \tag{63}$$

Here $L = l/\delta^2$ ($\nu D = 1$) has been chosen to pick up the lower-order terms μ and u^2 in the continuum limit and $p = \delta(\bar{p} - i\rho/\sqrt{\delta})$ has been rescaled accordingly; ellipses are higher orders in δ. Then

$$Z_1 \propto \text{constant}^L N\delta^3 \int_\mu^\infty dy \int_{i\rho/\sqrt{\delta}-\pi/\delta}^{i\rho/\sqrt{\delta}+\pi/\delta} d\bar{p}\, e^{-9\bar{p}^2 l - 3yl/2 + \mu l} + \text{regular}$$

$$= \tfrac{2}{3}\text{constant}^L N\delta^3 \frac{e^{-\mu l/2}}{l^{3/2}} + \text{regular}. \tag{64}$$

The finite-i region now also gives something analytic in μ, but in this case nevertheless represents *finite-$\mathcal{A}$* surfaces. The $\mathcal{A} = 0$ surfaces—$\mathcal{A} \sim O(\delta^2)$ more precisely—are characterized by the fact that a finite number of derivatives with respect to μ can remove their contribution, since each derivative brings down a factor of $\mathcal{A}$. The first term in (64) cannot be removed by differentiating. It can be interpreted as due to surfaces dense with polymer whose area is proportional to the polymer length, which is now conjugate to μ; for small l differentiating does remove its contribution. The result $a = -3/2$ and $\nu D = 1$ agrees with a dense phase of polymers constructed in [9]. This is discussed futher in the next section.

For $c < \sqrt{2} - 1$ the author has not been able arrive at a wholly satisfactory quantitative understanding of a scaling limit with fixed polymer length L. It was shown above that Z_1 no longer has an obvious universal contribution from the finite-i region. The precursor to this was already seen at $c = \sqrt{2} - 1$ where the finite-i region gave something regular in μ but nonetheless had an interpretation in terms of scaling surfaces and polymer. The origin of the difficulty seems to lie in the fact of working with a polymer of fixed length L which is dense on the surface. To resolve ambiguity and identify a part singular in μ it turns out

to be necessary to sum over L with respect to a monomer fugacity K. This will be done in the next section in the more general context of scaling operator two-point functions.

One notes, however, that the polymer in the low-temperature phase should to be in an ultra-compact state due to the allowance of multiple occupation of links in the model. This is highlighted in the low-temperature expansion; expanding (41) about $c = 0$, only a finite number of terms survive to given order of c. The result is

$$Z_1' = \frac{(2L)!}{(L!)^2} \left[1 + \frac{c^2 L}{3} (1 - 2u\delta + O(\delta^2, 1/L)) + O(c^4 L^2) \right]. \tag{65}$$

These terms have a clear geometrical meaning. At $O(c^0)$ the term $(2L!)/(L!)^2 \sim 2^{2L}/\sqrt{L}$ corresponds to the number of ways of folding the polymer so that it has no connections into the surface at all. Each subsequent power of $c^2 L$ corresponds to breaking a polymer–polymer bond, of which there are L, and inserting the dangling bonds into the dual graph on a ϕ^2 propogator. For example, order c^2 corresponds to folding the polymer entirely onto *one* of the links of the square lattice; equivalent in the dual graph to insertion of a $\text{Tr}\,\phi^2$ puncture operator times a combinatorial factor to account for the number of ways of pairing the $2L - 2$ other bonds of the polymer. Indeed, $\frac{\partial}{\partial\mu}\langle\text{Tr}\,\phi^2\rangle = 1 - 2u\delta + O(\delta^2)$, where the leading non-analytic term corresponds to two punctures on the universal finite-$\mathcal{A}$ surfaces, one from the polymer and the other from $\partial/\partial\mu$ (compare with equation (42)).
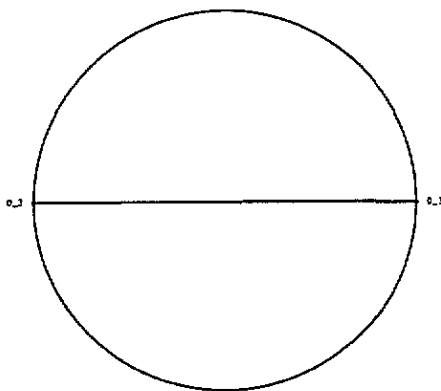

Figure 5. A watermelon network contributing to the two-point function $\langle\mathcal{O}_S\mathcal{O}_S\rangle$ for the particular case $S = 3$.

6. Scaling operators

In order to better appreciate the geometrical implications of the calculations just performed and convert the results to the corresponding ones on a fixed regular lattice it is expedient to derive the scaling dimensions of the usual polymer 'star' operators. These are the operators $\{\mathcal{O}_S\}$ which act as sources for S polymer lines; the two-point functions $\langle\mathcal{O}_S\mathcal{O}_S\rangle$ are watermelon networks illustrated in figure 5. These configurations are most simply discussed by introducing a variable K conjugate to the total length L of all polymers in the configuration and, in the case of a fluctuating lattice, the operators should be integrated over the lattice because there is no translational invariance. Thus one is considering the correlators

$$\left\langle \int \mathcal{O}_S \int \mathcal{O}_S \right\rangle = \frac{1}{Z^0} \sum_{L=1}^{\infty} K^L Z^S(L) \tag{66}$$

where $Z^S(L)$ is the partition function for the S-line watermelon network with fixed total length L of all the polymers. In particular for $S = 1$ one has $Z^1 = Z_1$ calculated in the previous section. In a dilute phase of polymers one also expects $Z^0 = Z_0$ (36) since the polymers have negligible backreaction on the background lattice. In a dense ($vD \leqslant 1$) phase, however, it is incorrect to normalize correlation functions with Z_0 since the dense polymer alters significantly the background geometry. In general one should normalize by a fluctuating lattice partition function with some scaling exponent $Z^0 \sim (\delta^2)^{2-\gamma_{str}}$ which takes into account any backreaction. Fortunately it can be determined by an independent argument [3]. The 'gravitational' scaling dimension Δ_S of $\mathcal{O}_S$ is given by the dependence on the lattice spacing

$$\left\langle \int \mathcal{O}_S \int \mathcal{O}_S \right\rangle \sim (\delta^2)^{2\Delta_S - 2}. \tag{67}$$

$\Delta > 1$ and $\Delta < 1$ imply irrelevance and relevance in the continuum limit respectively. These dimensions are related to the conformal weights $\Delta^{(0)}$ of operators in a conformal field theory in the plane [12] of central charge c by the KPZ formula [7]

$$\Delta^{(0)} = \Delta \left(1 - \frac{1 - \Delta}{\kappa} \right) \tag{68}$$

$$c = 1 - 6\frac{(1 - \kappa)^2}{\kappa} \qquad \kappa > 1. \tag{69}$$

In general the exponent γ_{str} does not determine c uniquely unless some further information is available. For the problem studied in this paper the charateristic behaviour of $\Delta^{(0)}$, that is whether it is negative, zero, relevant, marginal or irrelevant, follows that of the corresponding Δ.

The exponent Δ_1 is related to the power exponent in the single polymer partition function $Z_1 \sim e^{bL} L^a$

$$2\Delta_1 = \gamma_{str} - \frac{a + 1}{vD}. \tag{70}$$

Similarly $\Delta_1^{(0)}$ is related to the corresponding power exponent in the growth of the single polymer partition function in the plane, conventionally written $L^{\gamma-1}\mu^L$, by $\gamma/\nu = 2 - 4\Delta_1^{(0)}$ where the latter ν is the usual mean-square-size exponent $R^2 \sim L^{2\nu}$. Since $\mathcal{O}_2$ represents marking a point on the polymer, it couples to the length L and hence there exists are relation to νD given by [9]

$$\nu D = \frac{1}{1 - \Delta_2} \tag{71}$$

with a similar relation in the plane on setting $D = 2$. The higher even operators $\mathcal{O}_{2n}$ for $n > 1$ represent a contact between n different parts of the polymer simultaneously.

To illustrate the calculation of the star-operator two-point functions (66) consider the case $S = 2$. The relevant configurations are those of a closed loop of polymer with two marked points on the polymer. This constructed from two pieces of surface S_1 S_2 (figure 6), each with disc topology and the same boundary length L. The polymer is formed by first pinning together the two boundaries at (arbitrary) given points P_1 and P_2 on the boundary of S_1 and S_2 respectively, then sewing the disc boundaries together. Finally one chooses another (arbitrary) given point P_3 on the seam. The total amplitude for these configurations is $Z^2(L) = LZ_1(L/2)Z_1(L/2)$, that is two independent disc amplitudes, one with two marked points the other with one marked point on the boundary. Using the integral representations (47) and (56) one therefore has

$$Z^0 \left\langle \int \mathcal{O}_2 \int \mathcal{O}_2 \right\rangle = \frac{N^2 \delta^4}{(2\pi)^2} \int_0^\infty dz_1 \, dz_2 \int_{-\pi}^\pi dp_1 \, dp_2 \frac{1}{(1 - K\psi_1\psi_2)^2}. \tag{72}$$
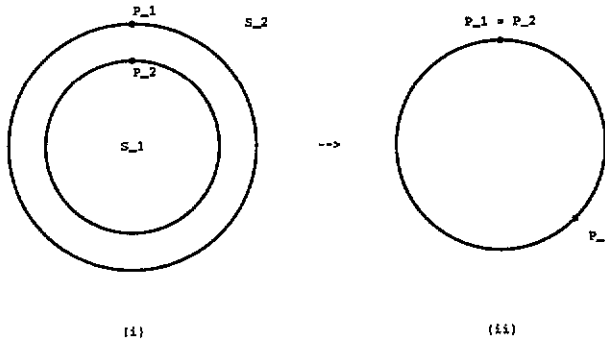
Figure 6. Constructing the configurations contributing to $\langle \mathcal{O}_2 \mathcal{O}_2 \rangle$. (i) Surfaces S_1 and S_2 have disc topology and the same boundary length (the figure is not to scale). (ii) Sewing the boundaries together gives a distinct polymer/surface configuration for each choice of points P_1, P_2, P_3.

Generalizing the construction to sewing S discs together to form the watermelon network (figure 5) gives

$$Z^0 \left\langle \int \mathcal{O}_S \int \mathcal{O}_S \right\rangle = \left(\frac{N \delta^2}{2\pi} \right)^S \int_{-\pi}^{\pi} dp_1 \dots dp_S \int_0^{\infty} dz_1 \dots dz_S \frac{1}{(1 - K \psi_1 \psi_2)} \frac{1}{(1 - K \psi_2 \psi_3)}$$

$$\dots \frac{1}{(1 - K \psi_S \psi_1)} \tag{73}$$

where hereafter the rescaling $\sqrt{1 - c^2} \psi \to \psi$ is used for convenience. By hypothesis this scales as $(\delta^2)^{2\Delta_S - \gamma_{str}}$. The critical regime for the number of monomers is achieved by tuning $K = K_c(1 - 2\delta^{2/\nu D} \sigma)$ where $1/\sqrt{K_c}$ is the critical (maximal) value of ψ which makes the denominators vanish in (73) to lowest order in δ. This singular behaviour of (73) allows one to read off the scaling dimension. It will now be computed for the various phases in turn.

(i) High-temperature phase..

From the analysis of the previous section (52) and (58), at least for the limits $c \to 1$ and $c \to (\sqrt{2} - 1)^+$, the scaling behaviour $p \sim \sqrt{\delta}$ and $z \sim \delta^2$ is appropriate and each denominator in (73) is $O(p^2, \sqrt{z + \mu}, \sigma)$. By power counting one has

$$2\Delta_S - \gamma_{str} = \tfrac{3}{4} S \tag{74}$$

as occurs at infinite temperature [9]. From (71) and $\nu D = 2$ one deduces $\gamma_{str} = -1/2$, which is known to be consistent with a conformal field theory central charge $c = 0$. Hence from the dimensions in the plane (68) $\Delta_S^{(0)} = (9S^2 - 4)/96$ one finds the usual dilute phase of self-avoiding random walks. It seems logical to assume that this behaviour is valid for the entire range $1 \geqslant c > \sqrt{2} - 1$. Note that Δ_1 is positive, meaning that the two ends of a single polymer attract, while Δ_2 is relevant but positive, charateristic of a dilute regime. The contact operators $\mathcal{O}_{2n}$, $n > 1$, are all irrelevant.

(ii) Collapse temperature.

At $c = \sqrt{2} - 1$ the scaling behaviour $p \sim \delta$ and $z \sim \delta^2$ was appropriate and the denominators are $O(p^2, z, \mu, \sigma)$. By power counting one finds

$$2\Delta_S = \gamma_{str} + \tfrac{1}{2} S \tag{75}$$

while $\nu D = 1$ implies $\gamma_{str} = -1$ consistent with the $c = -2$ conformal field theory at the end of the unitary minimal series (though it is not unitary). The scaling dimensions

$\Delta_S^{(0)} = (S^2 - 4)/16$ are those of the usual dense phase of self-avoiding random walks in the plane for which $R^2 \sim L$ [8]. $\mathcal{O}_1$ has negative dimension since the dense polymer screens the endpoints causing them to repel, while $\mathcal{O}_2$ is of dimension zero, like the identity operator characteristic of a dense phase. Of the contact operators, $\mathcal{O}_4$ is relevant and $\mathcal{O}_6$ marginal.

(iii) Low-temperature phase.

Consider perturbing $c = \sqrt{2} - 1 + \epsilon$ for small $\epsilon < 0$ now. Starting with $S = 1$ one recalls from (60) that the z-integral does not converge unless higher powers of u and μ in the expansion (48) are taken into account, in which case there is some saddle point at $z \sim O(1/\delta^2)$ corresponding to values of the discrete variable $i \sim O(N)$. This is quite odd behaviour but one can formally proceed perturbatively in ϵ. The first corrections have already been worked out in the exponent of (63). Collecting all the significant terms at leading non-zero order in ϵ

$$\psi \equiv a_1[\psi_0 - \delta^{2/\nu D} 9\bar{p}^2 - \alpha\epsilon^3\sqrt{z + \mu}\delta + (\mu - 3(\mu + z)/2)\delta^2 + \cdots] \quad (76)$$

where the constant part $\psi_0 = 1 + O(\epsilon^2)$. If $\epsilon < 0$ however, $a_1\psi_0$ is now longer than the maximal value of ψ (which determines K_c). To leading order in ϵ, the maximal value now occurs not for $z \to 0$ but for $z = z_c \sim O(\epsilon^6/\delta^2)$ (the term of order u^2 is now comparable to the $\epsilon^3 u$ term). The ellipses in (76) correspond to higher-order terms in δ and ϵ taking this fact into account. In detail, perturbing about this maximum as $z = z_c + \epsilon^3\bar{z}/\delta$ one finds $z_c = \alpha^2\epsilon^6/9\delta^2$ and Gaussian fluctuations

$$\psi \equiv a_1\left[\psi_0 + \frac{\alpha^2\epsilon^6}{6} - 9\delta^2\bar{p}^2 + \mu\delta^2 - \frac{27\bar{z}^2\delta^2}{8\alpha^2} + \cdots\right]. \quad (77)$$

The scaling laws for $\bar{p}$ and $\bar{z}$ in the last expression have been fixed to pick up the lowest order in μ, and $\nu D = 1$ as a result. The singular part of the $S = 1$ correlator is then

$$Z^0\left\langle \int \mathcal{O}_1 \int \mathcal{O}_1 \right\rangle \sim \epsilon^3 \int_{-\infty}^{+\infty} d\bar{p}\, d\bar{z} \frac{1}{\sigma + 9\bar{p}^2 - \mu + \frac{27\bar{z}^2}{8\alpha^2}} \quad (78)$$

$$\sim \epsilon^3 \log(\sigma - \mu) \qquad \sigma - \mu \to 0. \quad (79)$$

Similarly for $S > 1$ one has singular multiple integrals

$$Z^0\left\langle \int \mathcal{O}_S \int \mathcal{O}_S \right\rangle \sim (\epsilon^3)^S \int_{-\infty}^{+\infty} d\bar{p}_1 \ldots d\bar{p}_S\, d\bar{z}_1 \ldots d\bar{z}_S \frac{1}{2\sigma - 2\mu + 9\bar{p}_1^2 + \frac{27\bar{z}_1^2}{8\alpha^2} + 9\bar{p}_2^2 + \frac{27\bar{z}_2^2}{8\alpha^2}}$$

$$\cdots \frac{1}{2\sigma - 2\mu + 9\bar{p}_S^2 + \frac{27\bar{z}_S^2}{8\alpha^2} + 9\bar{p}_1^2 + \frac{27\bar{z}_1^2}{8\alpha^2}} \quad (80)$$

$$\sim \epsilon^{3S} \log(\sigma - \mu) \qquad \sigma - \mu \to 0. \quad (81)$$

This implies $\Delta_S = 0$ for all S and $\gamma_{str} = 0$. Logarithmic scaling violations are known to occur in the $c = 1$ conformal field theory of a free scalar field when coupled to two-dimensional quantum gravity precisely because of the fluctuating metric tensor. However, there is no obvious reason to expect this assignment of central charge in the present case; $\gamma_{str} = 0$ does not determine it uniquely. In fact the KPZ relation (68) would imply $\Delta_S^{(0)} = 0$ independent of c. This means that all operators scale like the identity operator. In particular $\Delta_1^{(0)} = 0$ means that the polymer endpoints move freely, the repulsion due to dense polymer being compensated by the attraction due to the tendency of the polymer to collapse. The result $\Delta_{2n}^{(0)} = 0$, $n > 1$, would seem to imply a dense regime for both monomers and contacts between monomers—the lattice is filled many times by the polymer. To this leading order in ϵ at least the multiple covering of the lattice by the polymer is not enough to change the ν-exponent from its usual dense-phase value $R^2 \sim L$ however. It seems plausible that the behaviour to first order in ϵ should extend a finite distance into the low-temperature phase but this remains a conjecture.

7. Conclusions

To summarize the results, using random matrices to solve the polymer folding model on a fluctuating two-dimensional lattice and then translating the results to the regular lattice using KPZ scaling, a collapse transition was found from a dilute to dense regime. On the fluctuating lattice the transition was third order. Although there are no rigorous bounds, all known examples of bulk phase transitions show up with lower order on the regular lattice; the collapse transition of a dilute polymer in two dimensions is really like a boundary phase transition but nevertheless one might guess that it is first or second order. While the high-temperature phase and collapse point correspond to the usual dilute and dense phases of self-avoiding walks in the plane respectively, the understanding of the low-temperature phase is still incomplete. Some arguments were given to suggest an ultra-compact regime in which the polymer covers the lattice many times due to the possibility of multiple occupation of links. This latter feature is a particularly unphysical aspect of the model as regards real polymers and results in a different universality class of collapse transition from models where multiple occupation of links is forbidden. The reformulation of such models in terms of percolating clusters [3] showed a Θ-point with behaviour intermediate between that of the standard dilute and dense phases of self-avoiding walks.

The indirect technique of solving discrete statistical models on fluctuating lattices by using random matrices and quantum gravity in order to predict the critical phenomena on regular lattices has received little attention before in the literature. While a little circuitous, the techniques invloved are relatively straightforward, having been developed by high-energy physicists in other contexts. It would be interesting to see how far one could push it in the condensed matter context. There are various possible extensions to the present work: it is easy to find (but not so easy to solve) random matrix models which forbid multiple occupation of links; this paper set up the formalism for a finite polymer fugacity Γ and it would be interesting to study the interplay of inter- and intra-polymer interactions; the polymer can also be given more exotic monomer content by using more complicated matrix models. These extensions are presently under investigation.

Acknowledgments

Appendix

Consider the expectation

$$\langle (1 - c^2)^L \, \mathrm{Tr}[(\phi + \alpha \psi)^{2L}] \rangle_{\Gamma=0} \tag{1}$$

with respect to the potential $V(\phi, \psi)$ (9). Expanding this operator in α

$$\mathrm{Tr}[(\phi + \alpha \psi)^{2L}] = \mathrm{Tr}[\phi^{2L} + \alpha(\phi^{2L-1}\psi + \phi^{2L-2}\psi\phi + \cdots)$$
$$+ \alpha^2(\phi^{2L-2}\psi^2 + \phi^{2L-3}\psi\phi\psi + \phi^{2L-3}\psi\psi\phi + \cdots) + \cdots + \alpha^{2L}\psi^{2L}] \tag{2}$$

one sees that this represents a co-polymer with random sequencing of monomers of type ψ and type ϕ, the latter being made out of the same material as the background lattice as it were, together with a fugacity α for the proportion which are type ψ. The weights of V are such that monomers of the same type tend to attract one another while monomers

of different types tend to repel. Unfortunately this is the opposite situation to that of the physically interesting one when monomers are electrically charged for example, but it can be solved immediately by the change of variable $\alpha\psi + \phi = \psi'$, $\phi = \phi'$. After some rescalings the new potential $V(\phi', \psi')$ is of the same form as the old one with the change

$$c \rightarrow \frac{\alpha c + 1}{\sqrt{\alpha^2 + 1 + 2\alpha c}}. \tag{3}$$

Therefore (1) can be evaluated using the results of the paper; in particular there is a collapse transition the temperature of which reaches zero when $\alpha^2 = -1 + 1/(\sqrt{2} - 1)^2$, there being no collapse for α smaller than this value.

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